Infinite dimensional covariance and non relativistic limits in time dependent theories

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Abstract

We give here some account of investigations for the possible role of infinite dimensional Lie algebras, whose simplest example is the classical Virasoro algebra, in time dependent systems or anisotropic statistical models. We expect a central extension to arise due to quantum corrections, but we first define the classical objects and so called "primary fields" transformation laws in order to be able to identify it precisely. The issue of non relativistic limit (negligible Compton wave length , or very big speed of light) of Minkovskian field theories is also of importance here.

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1 Introduction

Conformal theories, that is physical theories based on general relativity intuition (covariance laws, energy momentum tensor, differential symmetries), and using mathematical constructions such as Virasoro algebra and representation theory of infinite dimensional algebra, have a great predictive power in modern description of critical two-dimensional phenomena. Such domains not only include statistical physics 2d models such as Ising and Potts models, but also quantum spin chains, or effective formulations such as rotationally invariant quantum theory around Kondo effect impurities, magnetic monopoles or black holes.
Very soon attempts have been made to extend these ideas and concepts such as central charge to a wider physical context, such as higher dimensional field theories at some renormalisation group fixed point.
Good news may come from another front, where in tackling anisotropic statistical models, or time dependent problems, M. Henkel recognized some classical Virasoro symmetry and infinite dimensional generalisation of Galileo transformation.
It is also an important problem to deal with a complete description of renormalisation group flow between various critical theories, using eventually operators similar to heat kernels.
We present here some first attempts in this new program, hoping to give more thorough computations, results and physical applications in the future. These include issues such as classical limit of relativistic theories, a topic discussed in a rather difficult paper by Barut.

2 General set up for differential operators

Proposition

On any d-manifold, if x is a chart, $B = \sum_{\mu=1}^{d} B^\mu(x)\partial_\mu$ is a non vanishing, smooth vector field, $C = C(x)$ is an integrable function on any smooth path, ρ is a small real number, then

$$\phi(x,\rho) = exp(\rho(B+C))\ \psi(x) \ = \ exp\Big(\mathcal{T}(x,\rho)\Big)\ \psi(x'(x,\rho)) \qquad (1)$$

where the local diffeomorphism x' doesn't depend on C, and $\mathcal{T}$ is explicitly given below.

Proof is geometric:
We define $x'(x,\rho)$ to be the flow of the vector field B, i.e. (in an appropriate neighbourhood of the point of coordinates x) the solution of

$$\frac{\partial x'^{\mu}}{\partial \rho} = B^{\mu}(x'(x,\rho)) \tag{2}$$

satisfying the initial condition $x'(x,\rho=0)=x$. On the other hand, definition of the $\mathcal{T}$ "phase" is:

$$\mathcal{T}(x,\rho) = \int_0^{\rho} C(x'(x,s))\; ds \tag{3}$$

ϕ is identified as the solution of

$$\frac{\partial \phi}{\partial \rho} = (B+C)\phi \tag{4}$$

satisfying the initial condition $\phi(x,0)=\psi(x)$. Differentiation with respect to ρ gives

$$lhs = (B+C)(x')\phi$$

whereas application of $(B+C)(x)$ to the r.h.s. gives an apparently different expression. In fact proving the above proposition amounts to establishing the
Key lemma: for any ν, and small enough ρ :

$$B^{\nu}(x'(x,\rho)) = \sum_{\mu} B^{\mu}(x)\; \frac{\partial x'^{\nu}}{\partial x^{\mu}}(x,\rho) \tag{5}$$

(This means the solution x' is such that B at the image is the transform of B at the origin , in the sense of change of coordinate systems formulae....)

Proof of the lemma:
Let H be an hypersurface containing x and such that $B(x)$ is not tangent to H. We restrict ourself to a domain of H containing x where $B(h)$ is not tangent to H. Then the map :

$$\begin{array}{rcl} H\times\mathbb{R} & \rightarrow & \mathbb{R}^d \\ (h,\delta) & \rightarrow & y(h,\delta) \quad \text{solution of} \quad \dfrac{\partial y^{\mu}}{\partial \delta} = B^{\mu}(y(h,\delta)) \\ & & \quad\quad \text{with initial condition} \quad y(h,\delta=0)=h \end{array}$$

is one to one in a neighbourhood of x , and its inverse $x \longrightarrow (h_x, \delta_x)$ allows us to define instead of x^μ a new coordinate system h^α :

$$h^0 = \delta \text{ , and } (h^\alpha),\ \alpha = 1, \ldots, d-1 \text{ are local coordinates on H} \tag{6}$$

This system is such that $x'(x, \rho) = y(h_x, \rho + \delta_x)$ for x and x' close enough to H.
A simple reasonning expressing Jacobian matrices between these two coordinate systems gives the lemma: $J_1 = \frac{Dy^\nu}{Dh^\beta}$

$$\text{has line } \nu \text{ made of } \Big(B^\nu(y),\ \frac{\partial y^\nu}{\partial h^\beta}\ ,\ \beta = 1, \ldots, d-1\Big)$$

Its inverse $J_2 = \frac{Dh^\alpha}{Dx^\nu}$ has line $\alpha = 0$ equal to $\frac{\partial \delta}{\partial x^\nu}$
and other elements equal to $\frac{\partial h^\alpha}{\partial x^\nu}$.

$$\begin{aligned}
\text{Therefore } 1 &= (J_2 J_1)^0{}_0 = \sum_{\nu=1}^{d} \frac{\partial \delta}{\partial x^\nu}(y)\, B^\nu(y) \\
0 &= (J_2 J_1)^\alpha{}_0 = \sum_{\nu=1}^{d} \frac{\partial h^\alpha}{\partial x^\nu}(y)\, B^\nu(y)
\end{aligned}$$

which we can use at $y = y(h_x, \delta_x) = x$, so that:

$$\begin{aligned}
&\sum_{\nu=1}^{d} B^\nu(x)\, \frac{\partial x'^\mu}{\partial x^\nu\ |_\rho} = \sum_{\nu=1}^{d} B^\nu(x)\, \frac{\partial}{\partial x^\nu}\Big(y^\mu(h_x\, , \rho + \delta_x)\Big) \\
&= \sum_{\nu=1}^{d} B^\nu(x) \Big(B^\mu(x'(x,\rho))\frac{\partial \delta_x}{\partial x^\nu} + \sum_{\alpha=1}^{d-1} \frac{\partial y^\mu(h, \rho+\delta)}{\partial h^\alpha}\, \frac{\partial h^\alpha(x)}{\partial x^\nu}\ \Big) \\
&= B^\mu(x')\, (J_2 J_1)^0_0 + \sum_{\alpha=1}^{d-1} \frac{\partial y^\mu}{\partial h^\alpha}\, (J_2 J_1)^\alpha_0 \\
&= B^\mu(x')
\end{aligned}$$

Another (direct) proof is possible by systematic use of Dirac distributions and Green's functions. For any differential operator $O(x)$:

$$\Big(e^{\rho\, O(x)}\ \psi\Big)(x) \quad = \quad e^{\rho\, O(x)} \int dy\ \delta(x-y)\psi(y)$$

$$
\begin{aligned}
&= \int dy\; G_\rho(x,y)\psi(y) && (7)\\
\text{where } G_\rho(x,y) &= e^{\rho\, O(x)}\; \delta(x-y) && (8)
\end{aligned}
$$

Therefore the above proposition amounts to:

$$
\begin{aligned}
G_\rho(x,y) &= \Big(exp \int_0^\rho\; C(x'(x,s))\, ds\Big)\; \delta(y - x'(x,\rho)) && (9)\\
&= \sum_n \frac{\rho^n}{n!}\; \Big(C(x) + B^\mu\, \frac{\partial}{\partial x^\mu}\Big)^n\; \delta(x-y) && (10)
\end{aligned}
$$

$\delta(y - x'(x,\rho))$ can be computed by use of:

$$
A^\mu(x) := x'^\mu(x,\rho) - x^\mu \; = \sum_{n=1}^{\infty}\; \frac{\rho^n}{n!}\; \Big((B\cdot\partial_x)^{n-1}B^\mu\Big)(x) \qquad (11)
$$

$$
\begin{aligned}
&\text{Thus}\;\; \delta(y - x'(x,\rho)) && (12)\\
&= \; \delta(y - A^\mu(x) - x) = \; exp\Big(-\; A^\mu(x)\frac{\partial}{\partial y^\mu}\Big)\; \delta(y-x) \\
&= \Big(1 + \rho B^\mu\partial_\mu + \frac{\rho^2}{2}\Big((B\cdot\partial B^\mu)\partial_\mu + B^\mu B^\nu\partial_\mu\partial_\nu\Big) + \ldots\Big)\delta(x-y) && (13)
\end{aligned}
$$

$$
\text{where}\;\; \partial_\mu = \frac{\partial}{\partial x^\mu}
$$

3 An interpretation of formulae for non relativistic limits

We present here a geometric interpretation of a paper by Barut[2] relating non relativistic limits and group contraction. A peculiar case of the above proposition reads:

$$
\begin{aligned}
\phi &= exp\Big(\rho f(t)\Big(\frac{2\pi\, mc^2}{h} + \partial_t\Big)\Big)\; \psi(t) \\
&= exp\; \Big(\frac{2\pi\, mc^2}{h}\Big(t'(t,\rho) - t\Big)\Big)\psi(t'(t,\rho)) && (14)
\end{aligned}
$$

A direct proof of this equation, can be easily set up by considering the function

$$t \;:=\; T(\delta) \text{ reciprocal of } \delta := \int^t \frac{du}{f(u)} \tag{15}$$

$$\text{then} \quad t' \;:=\; T(\delta+\rho) = t'(t,\rho) \tag{16}$$

$$\text{and} \quad \rho \;=\; \int_t^{t'} \frac{du}{f(u)} \quad , \quad d\rho = \frac{dt'}{f(t')} - \frac{dt}{f(t)} \tag{17}$$

We would like to point out a rigorous geometric way of reproducing some of Barut's results: set

$$x_0 := c\tau \ , \ \phi_t(x_0,x) \;:=\; exp\Big(\tau\Big(\frac{2\pi\, mc^2}{h} + \partial_t\Big)\Big)\ \psi(t,x) \tag{18}$$

$$= \; exp\Big(\frac{2\pi\, mc^2\tau}{h}\Big)\ \psi(t+\tau,x) \tag{19}$$

This functional relation is such that flat space Klein-Gordon operator applied to $\phi(x_0,x)$ is proportional in the non relativistic limit to diffusion one applied to $\psi(t,x)$. More precisely define:

$$\Psi_{x_0}(t,x) := exp\Big(\frac{2\pi\, mcx_0}{h}\Big)\psi\Big(t+\frac{x_0}{c}\ ,x\Big) = \phi_t(x_0,x) \tag{20}$$

If we consider in $\Psi_{x_0}(t,x)$ x_0 as a parameter, and t as a parameter in $\phi_t(x_0,x)$, we have:

$$\frac{\partial}{\partial x_0}\phi_t = \Big(\frac{2\pi\, mc}{h} + \frac{1}{c}\frac{\partial}{\partial t}\Big)\Psi_{x_0} \tag{21}$$

This identity expresses the so called "group contraction trick" which insures:

$$\Big(\partial_0^2 - \partial_x^2 - \frac{4\pi^2 m^2 c^2}{h^2}\Big) \quad \phi_t(x_0,x) = \Big(\frac{4\pi m}{h}\partial_t - \partial_x^2 + \frac{1}{c^2}\partial_t^2\Big)\ \Psi_{x_0}(t,x) \tag{22}$$

The last term of the r.h.s. being negligible in the following "non relativistic limit".

4 Space - Time covariance

The following differential operators, considered by M. Henkel in his book,

$$
\begin{aligned}
-X_\varepsilon &:= \varepsilon(t)\partial_t + \frac{N}{2}\dot{\varepsilon}(t)\ (\ r\partial_r + \chi) + \frac{mr^{2/N}}{4}\ddot{\varepsilon}(t) && (23)\\
&:= -\sum_{n\in\mathbb{Z}} \epsilon_n X_n && (24)\\
\text{with} \quad \varepsilon(t) &:= \sum_{n\in\mathbb{Z}} \varepsilon_n\, t^{n+1} && (25)
\end{aligned}
$$

satisfy classical Virasoro algebra:

$$
[X_\varepsilon, X_\eta] = X_{\dot{\varepsilon}\ \eta - \varepsilon\dot{\eta}} \tag{26}
$$

It is therefore a challenge to try using Virasoro covariance in the context of time dependent physics such as heat diffusion . A first step is derivation of some "primary field" transformation law:

Proposition

$$
\begin{aligned}
&\phi(t, r, \varepsilon) := exp(-X_\varepsilon)\psi(t, r) && (27)\\
&= \Big(\frac{\varepsilon(t')}{\varepsilon(t)}\Big)^{N\chi/2}\ exp\Big(\frac{m}{4}\Big(\frac{r'^{\ 2/N}\,\dot{\varepsilon}(t')}{\varepsilon(t')} - \frac{r^{2/N}\,\dot{\varepsilon}(t)}{\varepsilon(t)}\Big)\Big)\ \psi(t', r') && (28)\\
&r' = r'(t, r, \varepsilon) = r\ \Big(\frac{\varepsilon(t')}{\varepsilon(t)}\Big)^{N/2} && (29)
\end{aligned}
$$

$$
t' = t'(t, \varepsilon) \qquad \text{such that} \qquad 1 = \int_t^{t'} \frac{d\tau}{\varepsilon(\tau)} \tag{30}
$$

Proof of this law is obtained following the lines of the general method given in first section, note that t' here does not depend on $r,\ m,\ \chi, N$ and that

$$
\dddot{\varepsilon}(t'(\rho))\ \varepsilon(t'(\rho)) = \frac{d}{d\rho}\,\dot{\varepsilon}(t'(\rho)) \quad . \tag{31}
$$

Note that if we introduce some time dependent scale σ

$$
\varepsilon(t)\ := e^{\sigma(t)} \tag{32}
$$

$$
\begin{aligned}
&\phi(t,r) \quad dt^{\ n\chi/2} \quad exp\Big(\frac{mr^{2/n}}{4}\,\dot\sigma(t)\Big) \\
=\ &\psi(t',r') \quad dt'^{\ n\chi/2} \quad exp\Big(\frac{mr'^{\ 2/n}}{4}\,\dot\sigma(t')\Big)
\end{aligned} \tag{33}
$$

From now on we take $N = 2/\theta = 1$ in the language of statistical mechanics, this corresponds to the heat kernel situation.

A second step is to derive correlation in geometries related by such transformations: as an academic example suppose a two parameter diffeomorpism between flat space $(t,r) \in \mathbb{R}^d$ and half space:

$$
t' = T'\ exp(t/T) \qquad \frac{r'}{r} = \sqrt{\frac{T'}{T}}\ exp(t/2T\,) \tag{34}
$$

If $\phi(t,r)$ has propagator of a massless scalar of dimension $(d-2)/2$ we will obtain the prediction for:

$$
\begin{array}{cc}
0 < t' < \infty & < \psi(t',r')\psi(0,0) > = \\
\dfrac{1}{\Big(\frac{T}{t'}\,r'^2\ + T^2\,log^2\Big(\frac{t'}{T'}\Big)\Big)^{(d-2)/2}} & \Big(\dfrac{T}{t'}\Big)^{\chi/2}\ exp\Big(-\dfrac{mr'^2}{4t'}\Big)
\end{array} \tag{35}
$$

5 Energy-Momentum tensor, central charge

It is a conjectural program to try to generalise to the above situation the impressive achievements of conformal theories. Crucial concepts are the stress tensor, which is not really a tensor since it does not transform according to the homogeneous law of relativity but with an extra term proportional to the Schwarz derivative. The proportionality coefficient, called up to a rational number, central charge, measures the "number" of massless degrees of freedom of the theory. It is related to the celebrated trace anomaly and can be calculated in two dimensions by various computations, such as short distance expansions, partition functions evaluation by various regularisations, computation of finite size effect or spectrum of hamiltonians in conformally flat geometries. An account of such computations is given in [5, 7], as well as a discussion of relationship between these concepts in $d > 2$. Here the geometric point of view is important. For example the correct bosonic massless Weyl invariant action in any dimension d should include a term proportional

to $R\ \phi^2$. We therefore give in appendix some technicalities which will certainly reveal useful in this more subttle perspective.

6 Appendix 1 Riemann tensors

Let us give some technicalities, recently considered as fashionable in membranes literature. If a non degenerate metric $G_{MN} = g_{\mu\nu} \bigoplus h_{mn}$ is block diagonal in term of coordinates $X^M = (x^\mu,\, y^m)$, we have in Landau's conventions:

$$R_{\lambda\mu\sigma\nu} = \quad r_{\lambda\mu\sigma\nu}(x,\,(y)) \ + \frac{1}{4} h^{ab}\left(\partial_a g_{\mu\sigma}\,\partial_b g_{\nu\lambda} - \partial_a g_{\mu\nu}\,\partial_b g_{\lambda\sigma}\right) \tag{36}$$

where (y) means the coordinates y^m are considered as parameters ie r is the Riemann tensor relative to $g_{\mu,\nu}(x,y)$ but does not contain derivatives with respect to y. We also need all mixed tensors such as:

$$\begin{aligned} R_{\lambda\mu,\,sn} &= \tfrac{1}{4} \quad g^{\alpha\beta}\Big(\partial_s g_{\alpha\mu}\,\partial_n g_{\beta\lambda} - \partial_n g_{\alpha\mu}\,\partial_s g_{\beta\lambda}\Big) \\ &+ \tfrac{1}{4} \quad h^{ab}\Big(\partial_\mu h_{as}\partial_\lambda h_{bn} - \partial_\mu h_{an}\partial_\lambda h_{bs}\Big) \end{aligned} \tag{37}$$

$$\begin{aligned} R_{\mu\nu} \quad = \quad & r_{\mu\nu}(g(x,(y))) \\ & -\frac{1}{2}h^{ls}\Big(\partial_l\partial_s g_{\mu\nu} - \Gamma^a_{ls}(h)\,\partial_a g_{\mu\nu}\Big) \\ & +\frac{1}{2}g^{\alpha\beta}(dg_{\mu\alpha}\cdot dg_{\nu\beta}) - \frac{1}{4}\,(dg_{\mu\nu}\cdot dlog(g)) \\ & +\frac{1}{4}h^{ls}h^{ab}\,\partial_\mu h_{as}\partial_\nu h_{bl} \\ & -\frac{1}{2}h^{ls}\Big(\partial_\mu\partial_\nu h_{ls} - \Gamma^\alpha_{\mu\nu}(g)\partial_\alpha h_{ls}\Big) \end{aligned} \tag{38}$$

Useful notations are: $d_a log\ g\ = g^{\mu\nu}\,\partial_a g_{\mu\nu}$ and $df\cdot dj\ = h^{ab}(x,y)\,\partial_a f\,\partial_b j$, and similarly $\delta f\cdot\delta j = g^{\mu\nu}\,\partial_\mu f\partial_\nu j$.

$$\begin{aligned} R \ = \ & r(g(x^\mu\,,\,(y))) \ + \ r(h((x),\,y^m) \\ & -\frac{1}{4}((dlog\ g\ \cdot dlog\ g) + (\delta log\ h\cdot\delta log\ h)) \end{aligned}$$

$$
\begin{aligned}
&+\partial_a log\ g\, h^{mn}\, \Gamma^a_{mn}(h) \\
&+\partial_\alpha log\ h\ g^{\mu\nu}\, \Gamma^\alpha_{\mu\nu}(g) \\
&-g^{\mu\nu} h^{ab} (\partial_a \partial_b g_{\mu\nu} + \partial_\mu \partial_\nu h_{ab}) \\
&+\frac{3}{4} g^{\mu\nu} h^{ab} \left(g^{\alpha\beta}\, \partial_a g_{\mu\alpha} \partial_b g_{\nu\beta}\ + h^{mn}\, \partial_\alpha h_{am} \partial_\beta h_{bn} \right)
\end{aligned}
\tag{39}
$$

For a product of two manifolds, the scalar R is additive.

7 Appendix 2 Space and time in an accelerated laboratory

As a side application of mathematical methods used here we would like to bring attention to the fact that such formulae can be used for definition of physical space and time in an accelerated frame . This could be useful for the study of expanding universe, black hole physics (quantum melanodynamics), or of any accelerated matter system.

Physically, let's suppose we are in a rocket, or an errant planet, whose position of center of mass is $x = f(t)$ in a flat Minkovskian space time endowed with coordinate system (x, t).

For simplicity we'll write only one space coordinate; Coriolis forces could be considered in a further step. Classical arguments, given by Einstein, are that inside the moving object we dispose of a **physical** coordinate system x', t' , and everything happens similarly to what would happen in a locally inertial comoving frame. This means that we suppose the number x' characterizes a material point of our rocket, which is supposed (or kept) fixed. This is the case if $\frac{d^3 f}{d^3 t} = 0$ (we have in mind the case of a rocket launched with constant acceleration, which should be felt as equivalent to a gravitational field, and then following its trajectory at constant speed). In formulae, we therefore have locally:

$$
dx' \quad = \quad \frac{dx \ - \ v\ dt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{40}
$$

$$dt' = \frac{dt - \frac{v}{c^2}\,dx}{\sqrt{1-\frac{v^2}{c^2}}} \tag{41}$$

If speed v is constant, this system leads to the celebrated Lorentz equations for x' and t'. A point x outside is seen from the crew at x' = dilation factor $\times(x - vt)$. The Minkovski interval $ds^2 = (c^2 - v^2)\,dt^2 = c^2 d\tau^2$ being conserved, we have a proper time (at least at the center of mass $x' = 0$) flowing slowlier according to:

$$d\tau = dt' = \sqrt{1 - \frac{v^2(t)}{c^2}}\;dt \tag{42}$$

(This eq. follows from the above system if we have $dx' = 0$).
We would like to consider more carefully the situation where the speed is not a constant, and try to rigorously consider the above system as made of p.d.e.'s for diffeomorphisms $x'(t,x)$ and $t'(t,x)$.
We propose to consider as physical speed of a point labelled by x' the following quantity:
First, at any fixed t, invert $x' = x'(t,x)$ into $x = X(t,x')$ (note this is different from inverting (x,t) into (x',t')). Then define

$$v(t,x') := \frac{\partial X}{\partial t}(t,x') \tag{43}$$

Therefore in the above p.d.e.'s system we have both time and space dependent coefficients (that is necessary to avoid paradoxes) depending on:

$$v(t,x) := \frac{\partial X}{\partial t}(t,x'(t,x)) \tag{44}$$

Position of center of mass and proper time then appear as boundary conditions:

$$x'(t,f(t)) = 0 \tag{45}$$

$$t'(t,f(t)) = \tau(t) \tag{46}$$

8 Appendix 3 Geometric conformal theories

We would like to bring attention of the reader to some geometric interpretation of so called conformal invariance in Euclidean, critical, 2d statistical

(field) theories. "conformal transformations" are often considered as symmetries: Any experienced mathematical physicists should take this with grains of salt, because a meromorphic function $Z = f(z)$, e.g. $= z^n$ is not in general one to one and therefore is not an element of the group of diffeomorphism of some projective manifold. In fact this is related to the rich theory of ramified mappings, on which physical mathematics has also brought new enumerative results. Furthermore geometric concepts are important here. Two of these are the concepts of Riemann (also called normal or geodesic) coordinates, and Weyl rescaling of the Riemannian metric structure. A way of understanding conformal invariance is to consider the local diffeomorphism which expresses z', a normal coordinate after Weyl transformation in terms of z an old normal coordinate. This is explained and generalised to higher dimension in [5, 7, 10]

Acknowledgements

We thank for discussions M. Dubois-Violette, C. Godrèche, H. Hilhorst, M. Lüscher, Jean Bernard Zuber, P. Ruelle, as well as Francesco Ravanini and Romano Prodi for kind hospitality and nice atmosphere in Bologna.

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